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Low-frequency sound propagation in a quasi-one-dimensional flow

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A systematic low-frequency theory is developed for the propagation of one-dimensional sound waves in a variable-area duct. The mean flow in the duct is assumed to be isentropic, compressible, and one-dimensional. Two applications are made of the theory. One concerns the reflexion coefficient from a pipe–nozzle combination, in which case comparisons are also made with some experimental data. In the second application, we consider the case of a sonic throat separating subsonic and supersonic flow. In this case, if the mean Mach number distribution in addition to being unity at the throat is also stationary at the throat, there is an axial 'boundary-layer' region in which the impedance of the sound wave changes from a fundamentally unsteady (reflexion-free) value at the sonic throat to the quasi-steady value away from the throat.

1. Introduction

In the present paper, we consider the propagation of one-dimensional sound waves in a mean flow described by the equations of isentropic, one-dimensional, compressible flow. The equations governing such linear acoustic waves were given by Tsien (1952) and solutions to these equations have been recently studied by Marble & Candel (1977), Bohn (1977) and Davis & Johnson (1974). In Marble (1973), an analysis valid for vanishingly small frequency was developed for this class of problems. The purpose of the present analysis is to develop the $O(\beta)$ correction to Marble's work, where β is an appropriate non-dimensional frequency parameter. As with the WKB high-frequency theory, the solution to $O(\beta)$ for arbitrary mean flow distributions is reduced to a simple quadrature.

The motivation for this study arose originally from the discovery of an error in a treatment of this problem by Ffowcs Williams (1972). Having studied first the case where the mean flow distribution is entirely subsonic, the study was also extended to the case where a sonic throat is involved (a case considered by Tsien (1952) and Marble & Candel (1977)). If the axial derivative of the mean flow velocity is non-zero at the sonic throat, the low-frequency theory can be carried through in a straightforward manner. An interesting singular-perturbation problem arises, however, in the case of a sonic throat at low frequencies if the derivative of the axial velocity also vanishes at the sonic throat. In this case, the behaviour of the acoustic wave is fundamentally unsteady at the sonic throat but it must be quasi-steady a short distance away from the nozzle throat. It turns out that there is a 'boundary layer' of axial extent

$$O(\beta^{1/(n-1)}),$$

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where n is order of the lowest non-vanishing derivative of the axial velocity at the sonic throat ($n \geq 2$), over which the acoustic impedance of the sound wave undergoes a rapid transition. This boundary-layer behaviour cannot be anticipated from the work of Marble (1973). Neither can it be anticipated from the works of Marble & Candel (1977) or Tsien (1952) because these authors examined cases where the mean velocity distribution is assumed to be linear with axial distance so that, if the axial derivative of the mean velocity vanishes at one axial location, it vanishes everywhere and thus the *flow* degenerates to a uniform flow. It should be noted that, in the case of problems involving a sonic throat, the viewpoint of Tsien (1952) has been adopted (as in Marble & Candel 1977) regarding the acoustic impedance at the throat, namely that this impedance is assumed to have a value that ensures a bounded analytic solution at the throat.

Additionally, the utility of such quasi-one-dimensional acoustic calculations is illustrated by their ability to predict in some detail the experimental data of Imelmann (1978). While this agreement is encouraging, the analysis outlined in this paper is subject to the usual restrictions that pertain to all quasi-one-dimensional formulations of problems involving variable-area ducts.

2. Problem formulation and low-frequency solution

Consider a quasi-one-dimensional, isentropic, compressible steady flow in a variable-area duct. The steady pressure, density, temperature and axial velocity in the duct are given by the usual equations of quasi-one-dimensional flow provided the area variation is known. In the case of a sonic throat, we assume that an accelerating flow is being considered, i.e. upstream of the throat the flow is subsonic and downstream of the throat it is supersonic.

As shown by Tsien (1952), linear sound waves in such a flow are conveniently described by considering a non-dimensional pressure perturbation

$$p'/(\gamma p(x)) = \phi$$

and a non-dimensional velocity perturbation

$$\nu = u'/\overline{u}(x),$$

where ϕ, ν satisfy

$$d(\nu+\phi)/dx = j\omega\phi/\overline{u} \tag{1}$$

and

$$d(\overline{u}^2\nu+\overline{a}^2\phi)/dx = j\omega\overline{u}\nu, \tag{2}$$

where $\overline{a}$ is the steady speed of sound in the duct; $j = \sqrt{-1}$; p is the steady static pressure in the duct and p' the fluctuating (acoustic) pressure in the duct; $\overline{u}$ is the steady axial velocity in the duct and u' the fluctuating (acoustic) axial velocity in the duct; $\overline{x}$ is the axial co-ordinate, γ the specific heat ratio (taken as 1·4 in this paper) and ω the oscillation frequency in radians/second (all oscillations taken as $\exp(-j\omega t)$.

The equations (1), (2) can, of course, be readily integrated numerically, given initial values of ϕ, ν, and the several studies referred to earlier have in fact given many examples of such solutions. Since (1) and (2) are linear in ϕ, ν, the absolute values of ϕ, ν are not of much interest and the most important initial value usually specified at one axial location for the solution of (1), (2) is the ratio ϕ/ν, which may be termed the

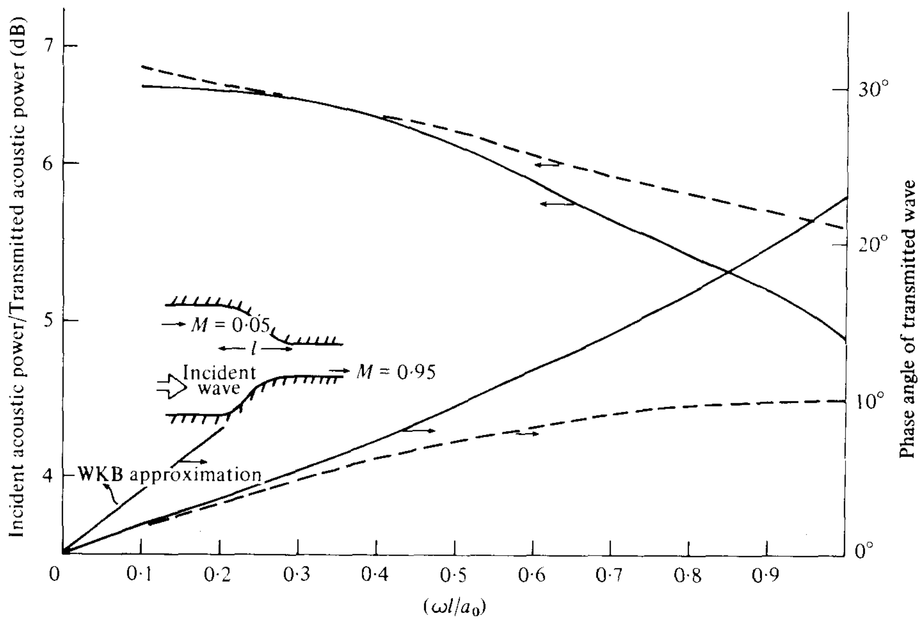

FIGURE 1. Sound transmission through a subsonic contraction. ——, exact; – – –, low-frequency approximation. a_0 is stagnation speed of sound.

impedance of the wave at one axial location. Specification of such an impedance at one location suffices to determine it everywhere else based on (1), (2).

For a low-frequency approximation, consider first the situation where the axial velocity distribution is subsonic everywhere. We non-dimensionalize all lengths by l, where l is a typical length scale over which a significant variation of the mean flow occurs (e.g. the duct length over which the area variation occurs), all velocities by a^*, the ideal fluid velocity at the sonic point. With x, u, a denoting non-dimensional quantities corresponding to $\bar{x}$, $\bar{u}$ and $\bar{a}$, (1) and (2) become

$$d(\nu+\phi)/dx = j\beta\phi/u \tag{3}$$

and

$$d(u^2\nu+a^2\phi)/dx = j\beta u\nu, \tag{4}$$

where

$$\beta = \omega l/a^*.$$

To construct the $O(\beta)$ approximation to (3), (4), we may proceed as follows. Let the ratio ϕ/ν be specified at one axial station, say $x = 0$. Since the absolute values of ϕ, ν are not pertinent, we assume (without loss of generality) that $(\phi+\nu) = 1$ at $x = 0$. Knowing ϕ/ν at $x = 0$, we can compute ϕ, ν at $x = 0$ as ϕ_i, ν_i. Then we first derive the $O(\beta^0)$ (Marble 1973) approximation to ϕ, ν at x by solving the pair of linear simultaneous equations

$$\nu+\phi = 1, \tag{5}$$

$$u^2\nu+a^2\phi = \{[u^2\nu_i+a^2\phi_i] \text{ at } x = 0\} = b. \tag{6}$$

Let the solution to (5), (6) be denoted by ϕ_0, ν_0. Also let

$$f(x) = \phi_0/(u(\phi_0+\nu_0)) = \phi_0/u$$

and $$g(x) = u\nu_0/(a^2\phi_0 + u^2\nu_0) = u\nu_0/b.$$

Rewriting (3) and (4) as

$$\frac{d}{dx}(\nu+\phi) = j\beta\frac{\phi}{(\nu+\phi)\,u}(\nu+\phi), \tag{7}$$

$$\frac{d}{dx}(a^2\phi+u^2\nu) = j\beta\frac{u\nu}{(a^2\phi+u^2\nu)}(a^2\phi+u^2\nu), \tag{8}$$

it is clear that a solution to (3) and (4) involving only quadratures and to $O(\beta)$ can be written as

$$\nu+\phi = \exp\left[j\beta\int_0^x f(\xi)\,d\xi\right], \tag{9}$$

$$u^2\nu+a^2\phi = b\exp\left[j\beta\int_0^x g(\xi)\,d\xi\right]. \tag{10}$$

Solving for ϕ, ν from (9), (10), we obtain a solution to $O(\beta)$ to (3), (4) satisfying the proper initial conditions. In (9) and (10), the exponential terms can be expanded to $O(\beta)$ as the approximation leading to (9), (10) is no more accurate than to $O(\beta)$.

In figure 1, the solution to an acoustic power transmission coefficient for the indicated flow is shown in amplitude (dB) and phase. Both the exact solution based on numerically integrating (3), (4) and the solution based on (9), (10) are shown. It is seen that, up to $\beta = 0{\cdot}6$, the low-frequency approximation of (9), (10) does a fair job of predicting the exact result. Clearly in the problem of figure 1, the ϕ/ν ratio would be taken specified at the downstream end at the reflexion-free value of M_d, where M_d is the Mach number of the mean flow at the downstream end. For these calculations, the steady axial velocity distribution is assumed to vary linearly between the upstream and downstream uniform flow regions. Also for reference purposes, the expected linear phase variation from a high-frequency theory (leading term of a WKB approximation) is shown.

3. Reflection coefficient for a low-frequency sound wave incident on a nozzle–jet flow termination

We can apply the above results also to a problem illustrated in figure 2. This problem was considered by Ffowcs Williams (1972) but his treatment requires some correction. Following him, we assume that the exit end of the pipe behaves as an 'open end', i.e. that the acoustic pressure vanishes at this end. There are two respects in which we wish to improve upon the treatment of Ffowcs Williams. Firstly, for the fluctuating acoustic velocity, Ffowcs Williams uses a zero-frequency and low-Mach-number approximation to (3). The solution to (5), (6) for $p' = 0$ at the open end yields that ϕ_0/ν_0 is $O(M^2)$ and hence Ffowcs Williams approximates (5) by $\nu_0 =$ constant and also uses this same approximation in integrating (8). There is, of course, no need to adopt either a low-Mach-number approximation or neglect $O(\beta)$ corrections to ν. Secondly, Ffowcs Williams linearizes a term $(\gamma p/((\gamma-1)\rho))$ as $(\gamma p'/((\gamma-1)\bar{\rho}))$ which is not correct because for acoustic waves $(\rho'/\bar{\rho}) = (p'/\bar{p})/\gamma$ and, hence, the correct linearized form of $(\gamma p/((\gamma-1)\rho))$ is just $(p'/\bar{p})$ (called $a^2\phi$ in the current analysis). Note that $\bar{\rho}$, ρ' denote the mean and fluctuating density and $\rho = \bar{\rho}+\rho'$.

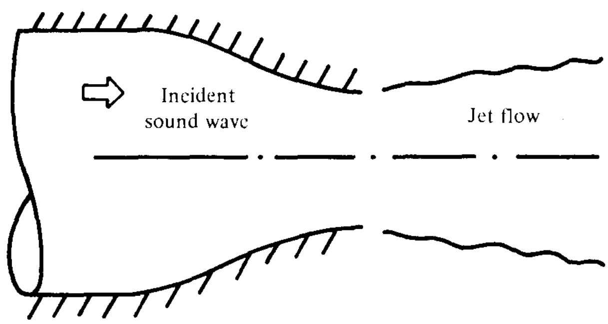

FIGURE 2. Reflexion of sound from a nozzle–jet flow termination.

A straightforward use of (5)–(10) (with $\phi = 0$) at the exit plane) yields the following expression for the reflexion coefficient (for the acoustic pressure). Let subscripts u, e be associated with upstream and exit quantities. Let M denote the Mach number of the steady flow and define two integrals of the steady flow as

$$I_1 = \int_0^1 \frac{u\,(a^2-u_e^2)}{u_e^2\,(a^2-u^2)}\,dx, \tag{11}$$

$$I_2 = \int_0^1 \frac{(u_e^2-u^2)}{u\,(a^2-u^2)}\,dx. \tag{12}$$

Also let $\theta_{1,2} = \beta I_{1,2}$. Then the reflexion coefficient, R, can be written as

$$-\left(\frac{1+M_u}{1-M_u}\right)\frac{[(a_u u_u - u_e^2)-j(\theta_2 a_u u_u - u_e^2\theta_1)]}{[(a_u u_u + u_e^2)-j(\theta_2 a_u u_u + u_e^2\theta_1)]}. \tag{13}$$

Bechert (1979) has pointed out one extraordinary consequence of (13). Considering the zero frequency limit of (13), we can see that, if the steady flow field is such that $(a_u u_u - u_e^2) = 0$, then at the upstream end (at vanishingly small frequencies) the nozzle–jet system will appear *anechoic* (i.e. $R = 0$). At low Mach numbers $(a_u u_u - u_e^2 = 0)$ would require that $M_e = A_e/A_u$, where A_e, A_u denote the exit and upstream flow areas. As Bechert (1979) further points out, Imelmann (1978) has verified experimentally this remarkable prediction of (13).

Pursuing this matter further, we show, in figure 3, a detailed theory–data comparison with the data of Imelmann (1978). The assumptions of the theoretical predictions are:

(i) The acoustic pressure fluctuation is assumed to be zero at the pipe exit.

(ii) The quasi-one-dimensional theory yields results as a function of β (taken in figure 3 as $\omega l/a_0$, where a_0 is the stagnation speed of sound). Imelmann (1978) gives results as a function of $(\omega R_t/a_0)$, where R_t is the tube radius. To compare the two it is suggested that we assume that $l = R_t$. The length l is the axial distance over which the interior parallel flow in the tube changes to the exterior parallel flow in the jet. This estimate may be justified as follows. The interior flow may be modelled as a sink flow induced by a sink at the centre of an axisymmetric pipe. Assuming incompressible, potential flow the velocity potential Φ can be expressed as

$$\Phi = u_u x + \sum_1^\infty \exp(\alpha_n x/R_t)\, C_n J_0(\alpha_n r/R_t),$$

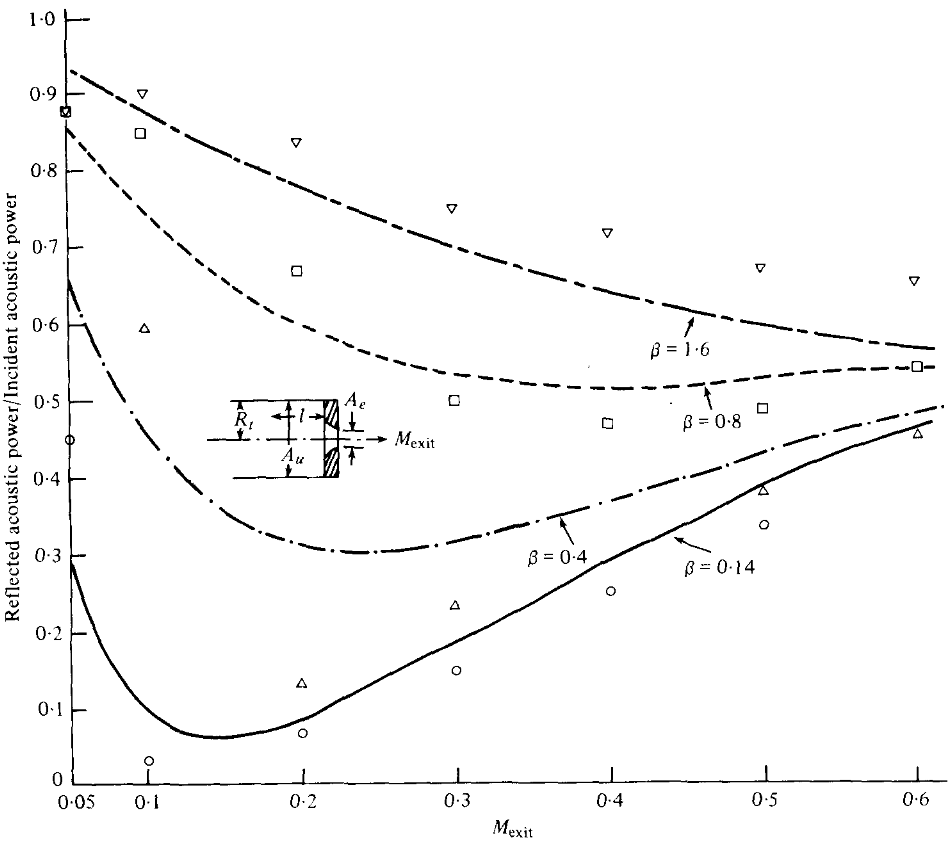

FIGURE 3. Acoustic reflexion coefficient from a pipe–nozzle flow. Data (Imelmann 1978) for the following $\omega R_t/a_0$: ○, 0·14; △, 0·4; □, 0·8; ▽, 1·6. The area ratio $A_e/A_u = 0{\cdot}132$.

where α_n are the successive zeros of $J_1(\alpha) = 0$ ($\alpha_1 = 3{\cdot}83$ and $\alpha_n > \alpha_1$ for $n \geqslant 2$). Hence over an axial distance equal to R_t, the non-parallel flow effects have decayed to about 2 % ($\exp(-3{\cdot}83)$) of their value at the pipe exit. Thus $l = R_t$ seems a reasonable estimate.

(iii) Since Imelmann's data covers a frequency range of more than 10 to 1, the theoretical curves shown in figure 3 were calculated by numerical integration of the exact linear system of (3), (4), assuming the steady axial velocity to vary linearly from inlet to exit. However, calculations based on the low-frequency approximation (13) were also carried out. These results were in better agreement with the data in figure 3 for $\beta = 0{\cdot}14$ and 0·4 than the results of the exact calculation and (as might be expected) in worse agreement for $\beta = 0{\cdot}8$ and 1·6.

It appears from figure 3 that the quasi-one-dimensional acoustic theory does an excellent job of predicting the variations of the reflexion coefficient with flow and frequency. If a data point falls within a factor of 1·26–0·79 of a prediction of an acoustic power ratio, the decibel error is within a dB and on this basis even the quantitative aspects of the theory–data agreement of figure 3 are most satisfactory. Apparently the assumption of zero exit acoustic pressure is quite reasonable. This result is perhaps due to the fact that $(\omega R_e/a_0)$ for the range of Imelmann's experiments is less than 0·582 and the Levine–Schwinger (1948) no flow, unflanged pipe solution would indicate $|R| \geqslant 0{\cdot}85$ at the pipe exit (zero acoustic pressure corresponds to $|R| = 1$) for such values of $(\omega R_e/a_0)$. Clearly the assumption of zero exit acoustic pressure will deteriorate for large values of β.

4. Converging–diverging nozzles with a sonic throat

This is the problem that has been considered in detail by Tsien (1952) and Marble & Candel (1977) for the special case when the steady axial velocity distribution in the nozzle is linear. For the rest of the paper, u' denotes du/dx and not the fluctuating acoustic velocity.

First in (3), (4) if we set $y = (\nu + \phi)$ it can be shown that y satisfies the linear, second-order equation

$$(1-M^2)\,u(uy')' = M^2\{(\gamma+1)\,uu'y' - 2j\beta uy' - 2j\beta u'y - \beta^2 y\}, \tag{14}$$

while (3) implies that

$$j\beta\phi = uy', \quad \nu = (y-\phi). \tag{15), (16}$$

(In (14)–(16), primes denote differentiation with respect to x.)

For (14), the location of the sonic throat ($M = 1$) constitutes at least a regular singular point (if $M' = 0$ at the sonic throat, the throat location is an irregular singular point) and Tsien's argument is that for a regular solution (ϕ/ν) must have a unique value at the sonic throat determined by setting $M = 1$, $u = 1$ in (14), (15), and (16) to yield

$$\frac{\phi}{\nu} = \frac{2u'(0) - j\beta}{(\gamma-1)\,u'(0) - j\beta} \tag{17}$$

at the sonic throat (assumed to be at $x = 0$).

On the other hand, consider the implications of the conservation laws for $(\nu+\phi)$, $(u^2\nu + a^2\phi)$ at zero frequency as expressed by (5) and (6) when a sonic throat is involved. At a sonic throat

$$(\nu+\phi) = (u^2\nu + a^2\phi)$$

(since $u = a = 1$ at the sonic throat) and hence (using subscripts zero to denote the zero frequency approximation)

$$\nu_0 + \phi_0 = u^2\nu_0 + a^2\phi_0 \quad \text{(everywhere)}. \tag{18}$$

Thus

$$\frac{\phi_0}{\nu_0} = \frac{(u^2-1)}{(1-a^2)} = \frac{2}{\gamma-1} \tag{19}$$

as long as u, $a \neq 1$ (i.e. away from the sonic throat) since the energy equation for the steady, isentropic flow is

$$\frac{u^2}{2} + \frac{a^2}{(\gamma-1)} = \frac{1}{2} + \frac{1}{(\gamma-1)}. \tag{20}$$

Comparison of (17) (which is an exact consequence of the existence of a regular solution at $M = 1$ ($x = 0$) of (14)) and (19) shows that *if* $u'(0)$ is of order unity and not a small quantity (in particular not zero), the zero frequency expansion of (17) gives the same value, $2(\gamma-1)^{-1}$, for ϕ/ν (to $O(\beta^0)$) as (19) (see Marble 1973).

Thus, if $u'(0) \neq 0$, the low-frequency expansion procedure for problems involving a sonic throat requires solution of a regular perturbation problem and differs little from the procedure already discussed except that the impedance ratio ϕ/ν must be taken as (17) specified at the sonic throat to ensure a regular solution. As Tsien (1952), Marble & Candel (1977) and Marble (1973) have noted, the upstream (subsonic) and downstream (supersonic) computations are uncoupled in this case, both being initiated from the sonic throat using (17). Normalizing the value of $(\phi+\nu)$ at the throat to be unity (without loss of generality), it is clear that the scheme of equations (5)–(10) can be adopted with

$$\phi_i = (2u'(0) - j\beta)/((\gamma+1)\,u'(0) - 2j\beta)$$

and

$$\nu_i = ((\gamma-1)\,u'(0) - j\beta)/((\gamma+1)\,u'(0) - 2j\beta).$$

Also

$$\phi_0 = 2/(\gamma+1) \quad \text{and} \quad \nu_0 = (\gamma-1)/(\gamma+1)$$

(independent of x).

In § 3 of Jacques (1975), it is suggested that 'Doppler contraction' effects will render inadmissible a low-frequency approximation near a sonic throat. On the other hand, Marble (1973) conjectured that such a restriction on the admissibility of a low-frequency approximation was 'probably over severe'. Since the $\beta = 0$ limit of (17) and (19) are perfectly compatible, the present results clarify that it is Marble's conjecture that is correct in general. There is one circumstance, however, in which Jacques' suggestion has some merit which we now address.

Consider now the rather more interesting case where, in addition to a sonic throat at $x = 0$, $u'(0)$ is also zero there. Even at low frequencies now, comparison of (17) (with $u'(0) = 0$) and (19) (which is certainly valid away from the sonic throat) reveals that (ϕ/ν) must change rapidly from the reflexion-free value of unity at the sonic throat to the quasi-steady value of $2(\gamma-1)^{-1}$ away from the throat. That the value of (ϕ/ν) must be unity (to ensure a regular solution) at the sonic throat in this case is made plausible from the argument that with $M' = 0$ simultaneously at $M = 1$ the sonic flow condition at $x = 0$ is more extensive than with $M' \neq 0$.

The appropriate *uniformly valid* low-frequency approximation for (ϕ/ν) in this case can be readily deduced from (14). The left-hand side of (14) (by (15)) can be written as

$$j\beta(1-M^2)\,u\phi' = \text{left-hand side of (14).} \tag{21}$$

Now, away from the sonic throat, the expression on the left-hand side in (21) is small because (at vanishingly small frequencies) $\phi \sim 2(\gamma-1)^{-1} \sim$ constant and hence $\phi' = 0$. Similarly close to the sonic throat, it is small because $(1-M^2)$ is small. Hence, we deduce that an appropriate low-frequency approximation can be found simply by taking the right-hand side of (14) to be zero and using (15), (16) to express ϕ, ν in terms of y. Thus we arrive at

$$\frac{\phi_0}{\nu_0} = \left(\frac{2u'(x) - j\beta}{(\gamma-1)\,u'(x) - j\beta}\right) \tag{22}$$

as the appropriate uniformly valid $O(\beta^0)$ estimate.

The expression (22) clearly exhibits the requisite boundary-layer behaviour. If $u^{(n)}(0)$ is the least non-vanishing derivative of $u(x)$ at $x = 0$ (with $n \geqslant 2$), (22) shows that the axial extent of the transition layer in which (ϕ/ν) changes from a reflexion-free to a quasi-steady value is of $O(\beta^{1/(n-1)})$.

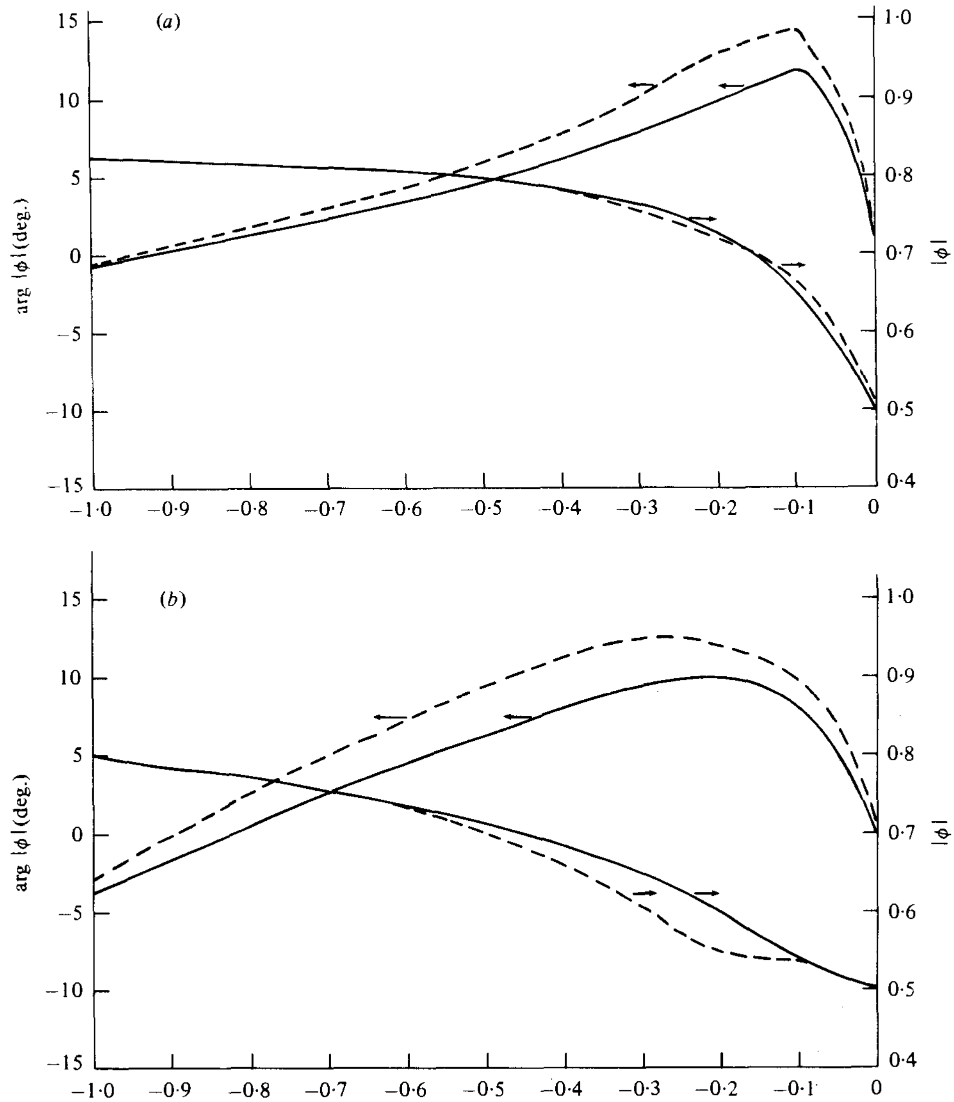

FIGURE 4. (*a*, *b*, *c*) Sound propagation in axisymmetric nozzle sketched in (*d*). (*a*) $\beta = 0{\cdot}1$, (*b*) $\beta = 0{\cdot}3$, (*c*) $\beta = 0{\cdot}75$. ——, exact; – – –, low-frequency approximation. (*d*) Axisymmetric nozzle radius corresponding to the calculations of (*a*), (*b*) and (*c*).

To compute the $O(\beta)$ correction to ϕ, ν we proceed now in the usual manner. Normalizing $(\phi+\nu)$ to be unity at $x = 0$ (sonic throat location), ϕ_i, ν_i would be 0·5 each (with $u'(0)$ at $x = 0$). ϕ_0, ν_0 would be

$$(2u'(x)-j\beta)/((\gamma+1)\,u'(x)-2j\beta)$$

and

$$((\gamma-1)\,u'(x)-j\beta)/((\gamma+1)\,u'(x)-2j\beta).$$

The solution to $O(\beta)$ would be obtained from (9), (10) with $b = 1$, $f(x) = \phi_0/u$ and $g(x) = u\nu_0$.

If this peculiar feature of the case with $u'(0) = 0$ at $M = 1$ is ignored (i.e. we take $\phi_0 = 2(\gamma+1)^{-1}$ and $\nu_0 = (\gamma-1)(\gamma+1)^{-1}$), two consequences ensue. Firstly, of course,

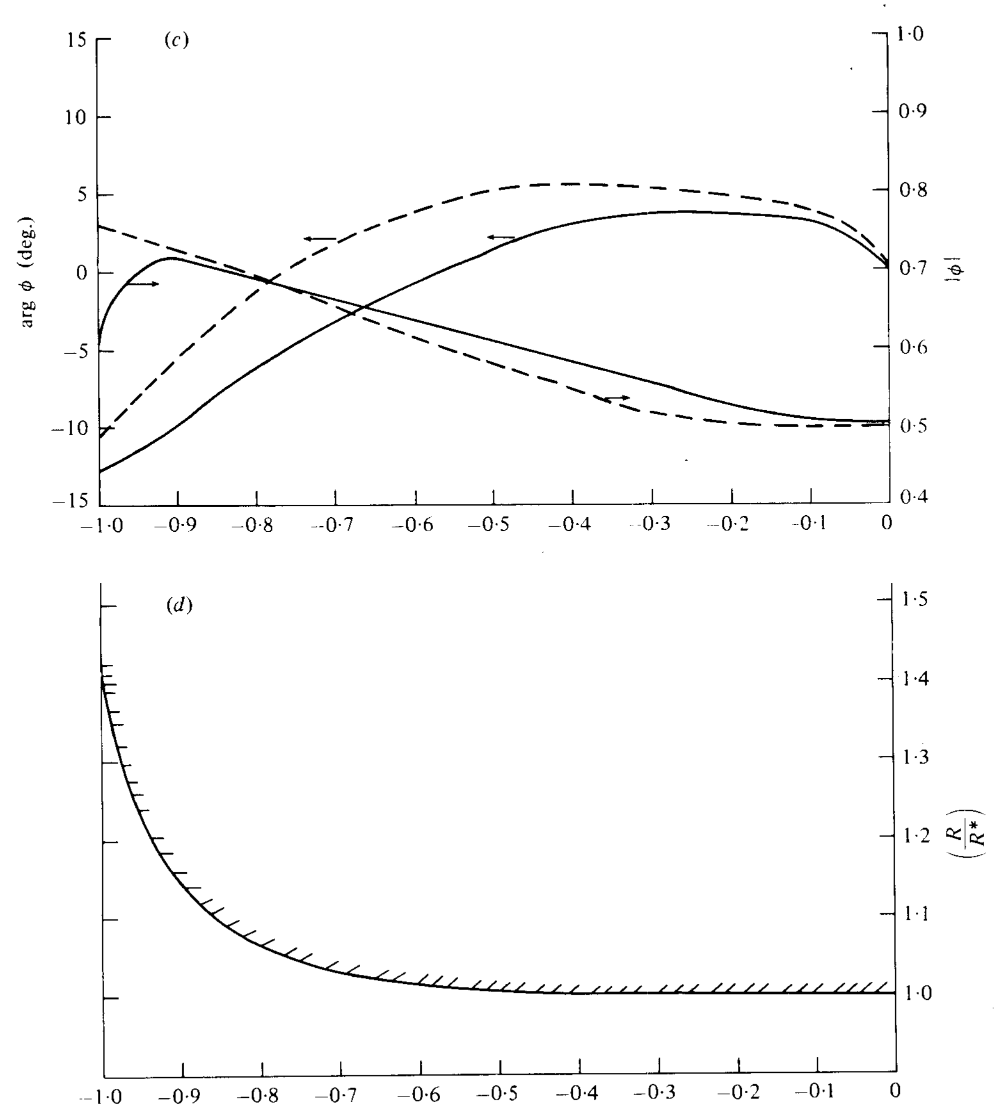

FIGURE 4(*c*, *d*). For legend see p. 89.

the boundary-layer region of $O(\beta^{1/(n-1)})$ in the zeroth-order solution would be missed. Secondly, however, precisely because the error in the zeroth-order solution is confined to a region of $O(\beta^{1/(n-1)})$, it is clear from (7), (8) that, in the case of computation of ϕ, ν away from the throat (e.g. in computing the reflexion coefficient of a sound wave incident from the subsonic side upon a sonic throat with $u'(0) = 0$), the error involved will be of $O(\beta^{n/(n-1)})$. Thus for instance, in cases of computation of such a reflexion coefficient, if $n = 2$, the error will be of $O(\beta^2)$, which is also the order of the error involved in using (22). However, if $n = 3$, whereas the approach based on (22) will still yield an error of $O(\beta^2)$, one based on (19) would yield an error of $O(\beta^{\frac{3}{2}})$. Thus for $n \geqslant 3$, an approach based on (22) rather than (19) in addition to yielding the boundary-layer structure also yields a better estimate (i.e. with an error of $O(\beta^2)$) for quantities such as a reflexion coefficient.

In figure 4, plots of ϕ in amplitude and phase (relative to the sonic-throat location) are shown for a nozzle whose Mach-number distribution is given by

$$M = (1+0{\cdot}91x^2\,\mathrm{sgn}\,(x))^{\frac{1}{2}}$$

for $\beta = 0{\cdot}1$, $0{\cdot}3$ and $0{\cdot}75$. Since the sonic throat uncouples the subsonic and supersonic regions of the flow, only the calculations for the subsonic region ($x \leqslant 0$) are shown. The low-frequency approximation based on (22) is also shown. Also the axisymmetric nozzle shape corresponding to the assumed Mach-number distribution is shown as part of figure 4. The 'exact' solution in figure 4 is based on seeking a regular solution to (14) by a numerical scheme.

The low-frequency approximation is seen to be adequate almost up to $\beta = 0{\cdot}75$. Both calculations show the transition region (which broadens with β) over which $|\phi|$ changes from $0{\cdot}5$ (at the sonic throat) to approximately $2(\gamma+1)^{-1}$ ($0{\cdot}83$ in the present case since γ is taken as $1{\cdot}4$ throughout this paper). The argument of ϕ at very low frequencies would be expected to be zero at both ends of the transition region.

5. Concluding remarks

The principal purpose of the present paper has been the development of a systematic theory to $O(\beta)$ for low-frequency sound propagation in a quasi-one-dimensional flow. The principal results are contained in equations (9), (10) to be used with (5), (6) or (19) or (22).

The two most noteworthy results of the study are the ability of quasi-one-dimensional acoustics to explain the data of Imelmann (1978) and the elucidation of the axial 'boundary layer' structure in the case of the derivative of the steady axial velocity vanishing at a sonic throat.

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